Examples of distribution compositions of technical object and system up states

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ABSTRACT: This article deals with compositions of distributions of technical object and system up states by combining exponential, Weibull, normal and gamma distributions. The compositions considered are those of two identical distributions with different parameters as well as compositions of different distributions. The analysis is illustrated with basic functional characteristics of the compositions. The data for these studies were collected from operational observations of failures in marine power plant systems.

1 INTRODUCTION

In classical problems of the theory of dependability the determination of probabilistic characteristics of object dependability is combined with searching for a model of the examined object's up time distribution. In this context the most common models are exponential, Weibull, normal, log-normal and gamma distributions. Sometimes the power distribution is used in cases when object up time has the upper bound, the constant b>0.

For large complex technical objects none of the mentioned distributions may prove good enough as a model of the object up time. The reson for this is that a number of various streams of failures of the examined object sum up and each of the streams may have completely different probabilistic characteristics. Various authors [Bob 1980, Bob 1977, Soł 1983] propose the extenstion of the various mathematical models of up time with *distribution compositions*.

To justify the proposal, we can compare the curve of empirical probability density obtained from field studies of the fuel oil installation and the sea water installation (Fig. 1 and 2) with the characteristics obtained from a simulation experiment for the composition of two distributions (Fig. 3). The installations observed were those of the marine power plant of ships owned by the Polish Steamship Company based in Szczecin [Prz 1997].

It can be observed that the probability density obtained for the distribution composition can assume a rather irregular form. This offers a chance to find a model appropriate for the distribution received from field studies.

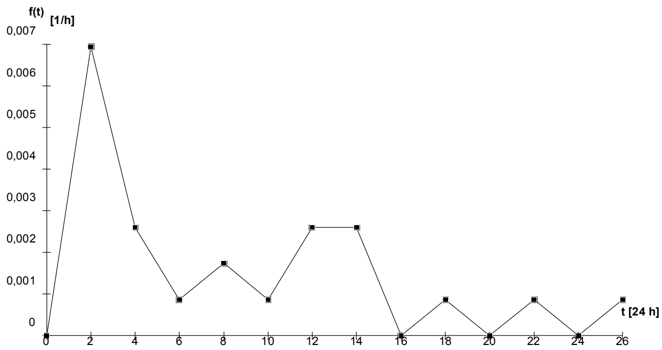

Figure 1. The characteristic of up time probability density for the fuel oil installation obtained from field studies of the ship 1 power plant.

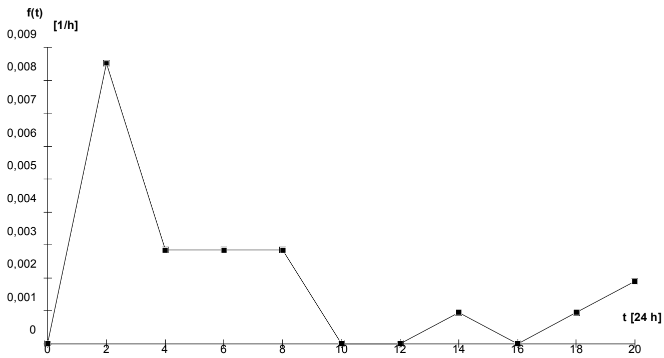

Figure 2. The characteristic of up time probability density for the sea water installation obtained from field studies of the ship 2 power plant.

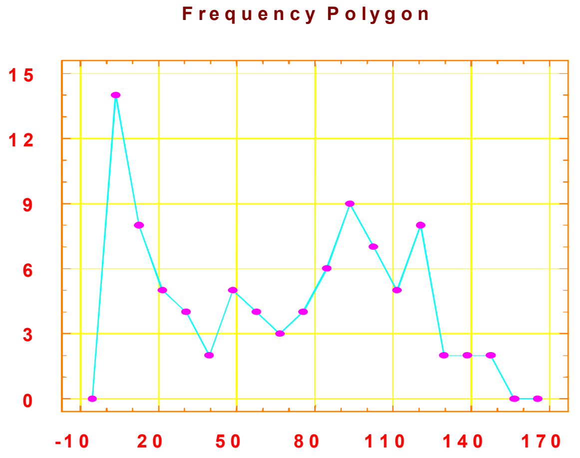

Figure 3. The probability density diagram obtained from a simulation experiment for a composition of two distributions.

F is a distribution function dependent on the parameter Θ and u is a certain probability density. In this case

$$W(x) = \int_{-\infty}^{\infty} F(x,\vartheta)u(\vartheta)d\vartheta \tag{1}$$

is the monotonic function of the variable x, increasing from 0 to 1, therefore it is a new distribution function. If F has the continuous density f, then W has a density written by this formula

$$w(x) = \int_{-\infty}^{\infty} f(x,\vartheta)u(\vartheta)d\vartheta \tag{2}$$

If the parameter Θ assumes the values $\vartheta_1, \vartheta_2, ...$ with the probabilities $p_1, p_2, ...,$ $p_k \geq 0, \sum_k p_k = 1$, then instead of integrating relative to the density u, we can sum up relative to the discrete distribution of the parameter Θ and then

$$w(x) = \sum_k f(x,\vartheta)p_k \tag{3}$$

determines a new probability density. This procedure is called *randomization*. The parameter ϑ is considered as the random variable, and the new probability distribution is determined on the plane *(x, Θ)*. Densities thus obtained are generally referred to as densities of mixed distributions. Such a definition of mixed distributions obtained through the randomisation of the parameter Θ results in many new distributions [Ana 1997b].

In further consideration, on the basis of the works [Bob 1980, Ana 1997a], we only focus on a particular case of distribution composition.

It is assumed below that random variables $X_1, X_2, ..., X_n$ are mixed. Their distribution functions are equal to, respectively, $F_1(x), F_2(x), ..., F_n(x)$, when the probability of encountering the term X_k is $p_k, \sum_k p_k = 1$.

Then the distribution function of the composition has this form

$$F(x) = \sum_k p_k F_k(x), \tag{4}$$

the composition probability density

$$f(x) = \sum_k p_k f_k(x) \tag{5}$$

the composition dependability function

$$R(x) = 1 - F(x) = \sum_k p_k[1 - F_k(x)] = \sum_k p_k R_k(x) \tag{6}$$

This analogy does not refer to the failure intensity, or to the leading function According to the definition, for the composition of n distributions the failure intensity takes this form

$$\lambda(x) = -\tfrac{d}{dx}[\ln R(x)] = -\tfrac{d}{dx}[\ln(\sum_k p_k R_k(x))] = \frac{\sum_k p_k f_k(x)}{\sum_k p_k R_k(x)} \tag{7}$$

and the leading function

$$\Lambda(x) = -\ln[R(x)] = -\ln[\sum_k p_k R_k(x)] = -\ln[1 - \sum_k p_k F_k(x)] \tag{8}$$

If $m_{k,r}$ denotes the ordinary moment of the r-th order for the k-th random variable X_k being part of the composition, then the ordinary moment of the r-th order for the whole composition has this form

$$m_r = \sum_k p_k m_{k,r}, \tag{9}$$

whereas the central moments of the r-th order have much more complex form

$$\mu_r = \int_{-\infty}^{\infty} (x - \sum_l p_l m_{l,1})^r \sum_k p_k f_k(x)dx \tag{10}$$

In some cases it is convenient to make use of the characteristic function of the distribution. For the composition of distributions the characteristic function has this form

$$\varphi(t) = \int_{-\infty}^{\infty} e^{itx} \sum_k p_k f_k(x)dx = \sum_k p_k \varphi_k(x) \tag{11}$$

where $\varphi_k(t)$ is the characteristic function of the random variable X_k $(k = 1, 2, ..., n)$.

$F_1, F_2, ...$ are distribution functions with $\varphi_1, \varphi_2, ...$ as their characteristic functions. If $p_k \geq 0$ and $\sum_k p_k = 1$, then the linear combination

$U=\sum_k p_k F_k$ is the distribution function with the characteristic function $\varphi=\sum_k p_k \varphi_k$.

As our task is to extend possible models of object up time distributions and to choose the best model for the data from operational shipboard observations, the models cannot be too complicated due to further statistical analysis. It seems that compositions limited to two random variables of different distributions on the one hand offer a sufficiently wide range of new models, on the other hand it makes it possible to obtain relatively simple functional and numerical characteristics of the considered models.

Some compositions of two different distributions are discussed below.

2 TWO WEIBULL DISTRIBUTIONS

The random variables X_1 and X_2 have Weibull distributions with the shape parameters, respectively, α_1 and α_2 and the scale parameters λ_1 and λ_2. It was assumed that $p(0<p<1)$ is the probability of encountering the random variable X_1. Then for the composition of these distributions we obtain:

the distribution function

$$F(x)=1-pe^{-\lambda_1 x^{\alpha_1}}-(1-p)e^{-\lambda_2 x^{\alpha_2}} \tag{12}$$

the probability density

$$f(x)=p\lambda_1\alpha_1 x^{\alpha_1-1}e^{-\lambda_1 x^{\alpha_1}}+(1-p)\lambda_2\alpha_2 x^{\alpha_2-1}e^{-\lambda_2 x^{\alpha_2}}, x\geq 0 \tag{13}$$

the leading function

$$R(x)=pe^{-\lambda_1 x^{\alpha_1}}+(1-p)e^{-\lambda_2 x^{\alpha_2}} \tag{14}$$

the failure intensity

$$\lambda(x)=\frac{p\lambda_1\alpha_1 x^{\alpha_1-1}e^{-\lambda_1 x^{\alpha_1}}+(1-p)\lambda_2\alpha_2 x^{\alpha_2-1}e^{-\lambda_2 x^{\alpha_2}}}{pe^{-\lambda_1 x^{\alpha_1}}+(1-p)e^{-\lambda_2 x^{\alpha_2}}} \tag{15}$$

and the leading function

$$\Lambda(x)=-\ln(pe^{-\lambda_1 x^{\alpha_1}}+(1-p)e^{-\lambda_2 x^{\alpha_2}}) \tag{16}$$

Numerical characteristics for this distribution are rather complex

$$\mu=EX=p\Gamma(1+\tfrac{1}{\alpha_1})\lambda_1^{-\frac{1}{\alpha_1}}+(1-p)\Gamma(1+\tfrac{1}{\alpha_2})\lambda_2^{-\frac{1}{\alpha_2}} \tag{17}$$

and the variance

$$\sigma^2=D^2X=p\lambda_1^{-\frac{2}{\alpha_1}}[\Gamma(1+\tfrac{2}{\alpha_1})-p\Gamma^2(1+\tfrac{1}{\alpha_1})]+$$
$$(1-p)\lambda_2^{-\frac{2}{\alpha_2}}[\Gamma(1+\tfrac{2}{\alpha_2})-(1-p)\Gamma^2(1+\tfrac{1}{\alpha_2})]-$$
$$-2p(1-p)\lambda_1^{-\frac{1}{\alpha_1}}\lambda_2^{-\frac{1}{\alpha_2}}\Gamma(1+\tfrac{1}{\alpha_1})\Gamma(1+\tfrac{1}{\alpha_2}) \tag{18}$$

The curves of basic functional characteristics of the Weibull distributions composition for $p=0.25, \lambda_1=0.1, \alpha_1=2. \lambda_2=0.02, \alpha_2=3$ are presented in Fig. 4 and 5.

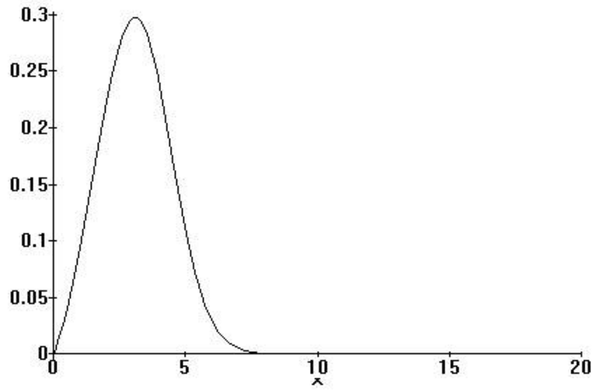

Figure 4. Probability density of the composition of two Weibull distributions.

Figure 5. Dependability function of the composition of two Weibull distributions.

3 TWO GAMMA DISTRIBUTIONS

The random variables X_1 and X_2 have gamma distributions with the parameters, respectively, (α_1,λ_1) and (α_2,λ_2) and p ($0<p<1$) is the probability of coming across the random variable X_1. Then for the composition of these distributions we obtain:

the probability density

$$f(x)=\frac{p\lambda_1^{\alpha_1}}{\Gamma(\alpha_1)}x^{\alpha_1-1}e^{-\lambda_1 x}+\frac{(1-p)\lambda_2^{\alpha_2}}{\Gamma(\alpha_2)}x^{\alpha_2-1}e^{-\lambda_2 x}, \quad x\geq 0 \tag{19}$$

the distribution function

$$F(x)=\frac{p\lambda_1^{\alpha_1}}{\Gamma(\alpha_1)}\int_0^x t^{\alpha_1-1}e^{-\lambda_1 t}dt+\frac{(1-p)\lambda_2^{\alpha_2}}{\Gamma(\alpha_2)}\int_0^x t^{\alpha_2-1}e^{-\lambda_2 t}dt \tag{20}$$

Too high complexity makes it impracticable to write down the function of failure intensity and the leading function for that distribution if we do not define the values of the parameters.

Fig. 6 and 7 below demonstrate curves of the density and the distribution function of the composition of two gamma distributions for the parameters $p = 0.25$, $\alpha_1 = 2, \lambda_1 = 0.1, \alpha_2 = 3,$ $\lambda_2 = 0.02$.

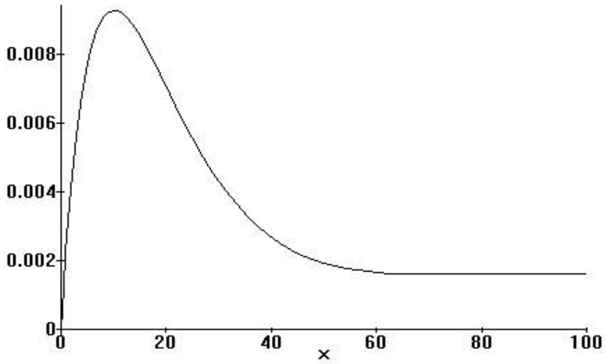

Figure 6. Probability density of the composition of two gamma distributions.

Figure 7. Distribution function of the composition of two gamma distributions.

The numerical characteristics for this distribution have this form:

The expected value

$$\mu = EX = \frac{p\alpha_1}{\lambda_1} + \frac{(1-p)\alpha_1}{\lambda_2} \tag{21}$$

and the variance

$$\sigma^2 = D^2 X = \frac{p\alpha_1[1+(1-p)\alpha_1]}{\lambda_1{}^2} + \frac{(1-p)\alpha_2(1+p\alpha_2)}{\lambda_2{}^2} - \frac{2p(1-p)\alpha_1\alpha_2}{\lambda_1\lambda_2} \tag{22}$$

4 TWO NORMAL DISTRIBUTIONS

The random variables X_1, X_2 have, respectively, the normal distributions $N(\mu_1,\sigma_1)$ and $N(\mu_2,\sigma_2)$, with $\mu_1 \geq 3\sigma_1$ and $\mu_2 \geq 3\sigma_2$, and p ($0<p<1$) is the probability of encountering the random variable X_1.

Because it is more convenient to use the density and standardized distribution function in the case of normal distributions ($\mu=0$, $\sigma=1$), the following denotations are introduced:

$$f_0(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right) \tag{23}$$

and

$$F_0(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left(-\frac{t^2}{2}\right)dt = \frac{1}{2} + \Phi(x) \tag{24}$$

where *Φ(x)* is Laplace function defined as follows

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{u^2}{2}}du \tag{25}$$

With these denotations, for the distribution composition we obtain:

the probability density

$$f(x) = \frac{p}{\sigma_1} f_0\left(\frac{x-\mu_1}{\sigma_1}\right) + \frac{1-p}{\sigma_2} f_0\left(\frac{x-\mu_2}{\sigma_2}\right) \tag{26}$$

the distribution function

$$F(x) = pF_0\left(\frac{x-\mu_1}{\sigma_1}\right) + (1-p)F_0\left(\frac{x-\mu_2}{\sigma_2}\right) \tag{27}$$

the dependability function

$$R(x) = 1 - pF_0\left(\frac{x-\mu_1}{\sigma_1}\right) - (1-p)F_0\left(\frac{x-\mu_2}{\sigma_2}\right) \tag{28}$$

the failure intensity

$$\lambda(x) = \frac{\frac{p}{\sigma_1} f_0\left(\frac{x-\mu_1}{\sigma_1}\right) + \frac{1-p}{\sigma_2} f_0\left(\frac{x-\mu_2}{\sigma_2}\right)}{1 - pF_0\left(\frac{x-\mu_1}{\sigma_1}\right) - (1-p)F_0\left(\frac{x-\mu_2}{\sigma_2}\right)} \tag{29}$$

the leading function

$$\Lambda(x) = -\ln\left[1 - pF_0\left(\frac{x-\mu_1}{\sigma_1}\right) - (1-p)F_0\left(\frac{x-\mu_2}{\sigma_2}\right)\right] \tag{30}$$

The expected value and variance of this distribution are as follows:

$$\mu = EX = p\mu_1 + (1-p)\mu_2 \tag{31}$$

$$\sigma^2 = D^2 X = p\sigma_1{}^2 + (1-p)\sigma_2{}^2 + p(1-p)(\mu_1-\mu_2)^2 \tag{32}$$

The random variables X_1 i X_2 have normal distributions truncated at zero with the parameters (μ_1,σ_1) and (μ_2,σ_2). Then for the composition of these distributions we obtain:

the distribution function

$$F_0(x) = p\frac{F_0(\frac{x-\mu_1}{\sigma_1}) - F_0(-\frac{\mu_1}{\sigma_1})}{1 - F_0(-\frac{\mu_1}{\sigma_1})} + (1-p)\frac{F_0(\frac{x-\mu_2}{\sigma_2}) - F_0(-\frac{\mu_2}{\sigma_2})}{1 - F_0(-\frac{\mu_2}{\sigma_2})} \quad (33)$$

the probability density

$$f_0(x) = \frac{pf_0(\frac{x-\mu_1}{\sigma_1})}{\sigma_1[1 - F_0(-\frac{\mu_1}{\sigma_1})]} + \frac{(1-p)f_0(\frac{x-\mu_2}{\sigma_2})}{\sigma_2[1 - F_0(-\frac{\mu_2}{\sigma_2})]} \quad (34)$$

The other characteristics of this distribution are not written down because in the general case their form is very complex.

Fig. 8 and 9 below present a comparison of the probability densities for various values of the parameters characterizing this composition.

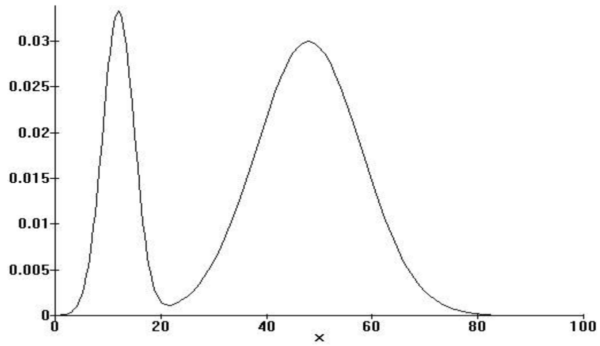

Figure 8. Probability density for the composition $N(12,3)$, $N(48,10)$, p=0.25.

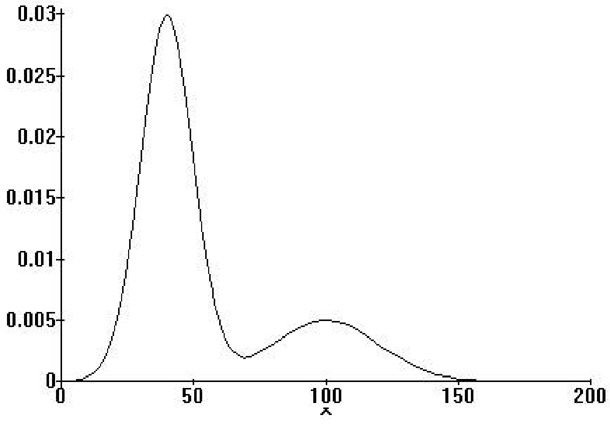

Figure 9. Probability density for the composition $N(40,10)$, $N(100,20)$, p=0.75.

5 THE EXPONENTIAL AND WEIBULL DISTRIBUTIONS

The random variable X_1 has the exponential distribution with the parameter λ_1, and the random variable X_2 has the Weibull distribution with the parameters α_2 and λ_2, and, as always, p ($0<p<1$) determines the probability of coming across the random variable X_1.

The following are obtained for the composition of these distributions:

The distribution function

$$F(x) = 1 - pe^{-\lambda_1 x} - (1-p)e^{-\lambda_2 x^{\alpha_2}} \quad (35)$$

the probability density

$$f(x) = p\lambda_1 e^{-\lambda_1 x} + (1-p)\lambda_2\alpha_2 x^{\alpha_2 - 1} e^{-\lambda_2 x^{\alpha_2}}, \quad \text{for } x \ge 0 \quad (36)$$

As the exponential distribution is a special case of the Weibull distribution, all the functional characteristics and numerical compositions of these distributions have the same form as that for the composition of two Weibull distributions (Chapter 2), assuming that $\alpha_1 = 1$.

The curves of probability density of that composition with different values of the parameters are shown in Fig. 10 and 11.

Figure 10. Probability density of the composition of exponential and Weibull distributions for $\lambda_1 = 0.01$, $\lambda_2 = 0.002$, $\alpha_2 = 3$, when p=0.25.

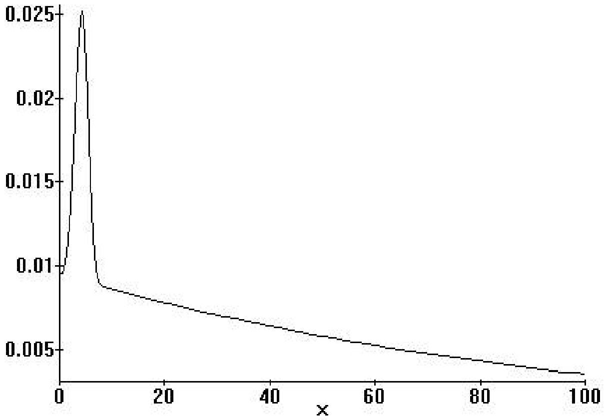

Figure 11. Probability density of the composition of exponential and Weibull distributions for $\lambda_1 = 0.01$, $\lambda_2 = 0.002$, $\alpha_2 = 4$, when p=0.95.

6 SUMMARY

As there is a large diversity of shapes of the curves representing probability densities of the distribution composition, the appropriate model of the up time of complex technical systems can be searched for among them in the case when none of the known distributions is good enough.

The problem to be solved is to find estimators of the parameters that are present in the compositions of distributions. The literature on the subject does not offer formulas for the relevant parameters of distribution compositions.

It can be noted that if the random variables $X_1, X_2, ..., X_n$ with the densities dependent on only one parameter $f_1(x,\vartheta_1), f_2(x,\vartheta_2), ..., f_n(x,\vartheta_n)$ are mixed, with the assumption that the random variable X_k is encountered with the probability p_k, $(\sum_k p_k = 1)$, then for an N-element random sample of the composition the likelihood function assumes this form

$$L(y_1, y_2, ..., y_N, \vartheta_1, \vartheta_2, ..., \vartheta_n) = \prod_{i=1}^{N}\left(\sum_k p_k f_k(y_i, \vartheta_k\right) \quad (37)$$

and its logarithm

$$\ln L(y_1, y_2, ..., y_N, \vartheta_1, \vartheta_2, ..., \vartheta_n) = \sum_{i=1}^{n}\ln\left(\sum_k p_k f_k(y_i, \vartheta_k)\right) \quad (38)$$

If we assume that the probabilities p_k are known, then taking advantage of the condition that there exists the maximum of the likelihood function logarithm, we obtain the system of n equations

$$\frac{\partial \ln L}{\partial \vartheta_r} = \sum_{i=1}^{N}\frac{\sum_k p_k \frac{\partial f_k(y_i,\vartheta_k)}{\partial \vartheta_r}}{\sum_k p_k f_k(y_i, \vartheta_k)} = 0 \quad \text{for} \quad r = 1, 2, ..., n \quad (39)$$

In order to determine the estimators of the parameters of this distribution composition the above system of equations has to be solved. In reality the situation gets even more complicated because the most frequently applied distributions have two parameters. Besides, the probabilities p_k are not known. Therefore, other statistical methods should be searched for to find an appropriate model of object up time from among the above discussed distribution compositions.

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